3101. It is given that $y = \mathrm{f}(x)$ has $x = k$ as a line of reflective symmetry, and also that

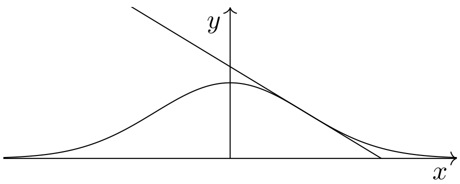
$$\int_0^k \mathrm{f}(x)\,dx = 1.$$

Evaluate the following integrals:

(a) $I_1 = \displaystyle\int_0^{2k} \mathrm{f}(x)\,dx$,

(b) $I_2 = \displaystyle\int_0^{2k} \mathrm{f}\left(\tfrac{1}{2}x\right)dx$.

3102. A bell-curve function is given as $\mathrm{f}(x) = e^{-\frac{1}{2}x^2}$.




Verify that the line $y\sqrt{e} = 2 - x$ is tangent to the bell curve $y = \mathrm{f}(x)$ at one of its points of inflection.

3103. The *Euler characteristic* of a polyhedron is defined by $\chi = V - E + F$, where $V, E, F$ are the numbers of vertices, edges and faces respectively.

Show that the Euler characteristics of the cube and the tetrahedron are the same.

3104. Four white counters and four black counters are put in a bag. Four counters are then drawn out. Find the probability of drawing out

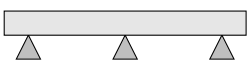
(a) two of each,

(b) three of one and one of the other.

3105. Points $P : (1, 1)$ and $Q : (1, -1)$ are labelled on the curve $x = y^2$. A tangent $T$ is drawn to the curve at $Q$, and a perpendicular is dropped to $T$ from $P$. Show that this perpendicular meets $T$ at the point $(1/5, -3/5)$.

3106. The graph $y = x^3 - x$ is transformed, by a stretch in the $x$ direction and a stretch in the $y$ direction, to the graph $y = x^3 - 8x$. Find the area scale factor associated with the transformation.

3107. A smooth, uniform, non-rigid beam of mass $m$ is resting, as shown, on three supports. The supports are placed symmetrically about the centre:



Prove that the forces at the supports cannot be determined without further information.

3108. A negative quartic function f has, for some $q \in \mathbb{R}$, $\mathrm{f}(q) = 0$, $\mathrm{f}'(q) = 0$, $\mathrm{f}''(q) > 0$. Show that $y = \mathrm{f}(x)$ intersects the $x$ axis exactly three times.

3109. True or false?

(a) $\sin x \equiv |\sin x|$,

(b) $\sin^2 x \equiv |\sin^2 x|$,

(c) $\sin^3 x \equiv |\sin^3 x|$.

3110. An integral equation is given as

$$\int \frac{2}{y}\,dx = y + c.$$

Solve to find a general solution for $y$ in terms of $x$.

3111. Find the set of values of the constant $k$ for which the following graphs have at least one intersection:

$$x^3 + y^3 = 1,$$
$$x + y = k.$$

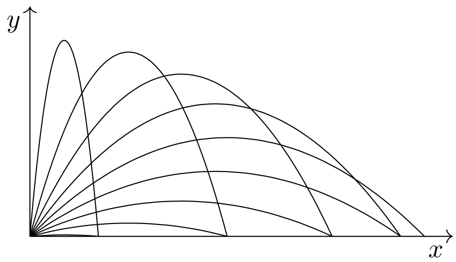
3112. Prove that, if $y = \mathrm{g}(x)$ has the $y$ axis as a line of symmetry, then $y = \mathrm{g}'(x)$ has rotational symmetry around the origin.

3113. The average value $\bar{y}$ of a function f on a domain $[a, b]$ is calculated using the formula

$$\bar{y} = \frac{1}{b-a}\int_a^b \mathrm{f}(x)\,dx.$$

Using a sketch and considering areas, explain the structure of the formula.

3114. A projectile is launched from ground level at speed $u$, at an angle randomly chosen on $[0, 90°]$.




Find the probability that the height $h$ attained by the particle will exceed $\frac{3u^2}{8g}$.

3115. A function has instruction

$$\mathrm{f} : x \mapsto \frac{x+1}{x^2+x+1}$$

(a) Find the largest real domain over which f is well defined.

(b) Find the range of f over this domain.

3116. It is given that the parabola $y = px^2 + qx + r$ has $x$ intercepts at $x = a, b$, and is stationary at $(u, v)$. Give the equation of the parabola with $x$ intercepts at $x = -a, -b$, which is stationary at $(-u, -v)$.

3117. If a polynomial f has $f'(0) = f'(1) = 2$, show that f cannot have degree 0 or 2.

3118. Express $a^9 + 5a^6 + 7a^3 + 3$ in terms of $(a^3 + 1)$.

3119. A cubic function f is defined as $f(x) = x^3 + x + k$, where $k < 0$ is a constant.

(a) Show that $f(x) = 0$ has exactly one root $\alpha$, and that $\alpha > 0$.

(b) Show that $(0, -k)$ is a point of inflection of the graph $y = f(x)$, and that the graph is convex for $x > 0$.

(c) Hence, explain, with reference to gradients, why the Newton-Raphson method, starting from $x_0 = 0$, will always succeed in finding the root $\alpha$.

3120. Show that the shortest path between the distinct sections of the graph $y(1 + x) = 1$ passes through the point $(-1, 0)$.

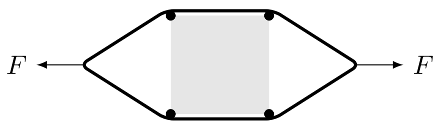
3121. (a) By writing $\cos 3x = \cos(2x + x)$ and using a compound-angle formula, show that

$$\cos x + 3\cos 3x \equiv 12\cos^3 x - 8\cos x.$$

(b) Hence, find the coordinates of all stationary points of $y = \sin x + \sin 3x$ for $x \in [0, \pi]$.

3122. A polynomial f is convex everywhere. Prove that f must be of even degree.

3123. A loop of light string of length 6 is passed around four smooth pegs, which lie at the vertices of a square of side length 1. The string is made taut by having two symmetrical forces of magnitude $F = 10\sqrt{3}$ N applied to it.




(a) Find the tension in the string.

(b) Show that each of the pegs experiences, due to the string, a force of $20\sin 15°$ N.

3124. The equation $x - \ln x - 2 = 0$ is not analytically solvable. This question concerns the use of fixed-point iteration to solve it numerically.

(a) Use a suitable rearrangement $x = g(x)$, where $g(x)$ has no asymptotes, to find the root near 3.1 to an accuracy of 4sf, establishing error bounds with a change of sign method.

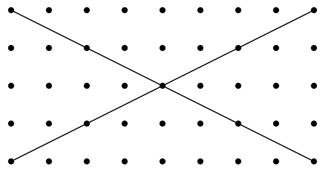
(b) Explain why your rearrangement cannot find the root near 0.1.

3125. A regular $n$-gon has vertices at $(\sqrt{3} + 1, \sqrt{3} - 1)$, $(\sqrt{6}, \sqrt{2})$ and $(2, 2)$. Determine $n$.

3126. Sketch $y = \dfrac{1}{1 + x^{2k}}$, for large $k \in \mathbb{N}$.

3127. Find $\displaystyle\int \cot 2x \operatorname{cosec} 2x \, dx$.

3128. Prove that, if an $m \times n$ lattice has diagonals which pass through at least one lattice point (other than the corners), then $m$ and $n$ have a common factor.



3129. A sample $\{x_i\}$ of size $n$, with mean $\bar{x}$ and standard deviation $s_x$, is transformed to a new sample $\{y_i\}$, according to the formula $y_i = ax_i^2 + b$, for some constants $a, b$. Show that, in terms of $n, \bar{x}, s_x, a, b$, it is

(a) possible to find an expression for $\bar{y}$, but

(b) not possible to find an expression for $s_y$.

3130. Find and classify all stationary points of the curve $y = \sin^2 x + \cos x$, and sketch it for $x \in [0, 2\pi]$.

3131. Solve the inequality $x^2 - \dfrac{1}{x} \geq 0$.

3132. Each of the sets $R$ and $S$ is defined as containing values simultaneously satisfying two inequalities:

$$R = \{x \in \mathbb{R} : f(x) \geq 0, g(x) \geq 0\},$$
$$S = \{x \in \mathbb{R} : f(x) \leq 0, g(x) \leq 0\}.$$

Find a counterexample to each of the following:

(a) $S \cup R = \mathbb{R}$,

(b) $S \cap R = \varnothing$.

3133. This question concerns the parabola $y = x^2$.

(a) Show that the equation of the normal to the parabola at $(p, p^2)$ is $y = -\frac{1}{2p}x + p^2 + \frac{1}{2}$.

(b) Hence, determine which of the points $(0, 2)$ and $(2, -0.5)$ is further from the parabola.

3134. Consider a regular polygon with $n$ sides, with $P$ as the perimeter and $R$ as the circumradius, i.e. the distance from the centre to the vertices.

(a) Show that $P = 2nR\sin\frac{180°}{n}$.

(b) Hence, prove that $\displaystyle\lim_{n\to\infty} n\sin\frac{180°}{n} = \pi$.

3135. Sketch $(3x - 2)^6 y = 1$.

3136. The function f has instruction

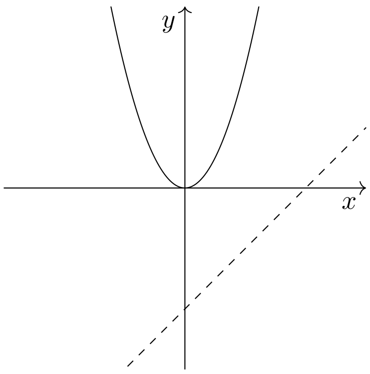
$$\mathrm{f} : x \mapsto \frac{x^4 + x^2 - 5}{x^4 + x^2 + 5}$$

Show that f may be assigned the domain $\mathbb{R}$.

3137. Scores $X$ and $Y$ on two tests are combined to give a total mark by a weighted average $M = aX + bY$, where $a, b > 0$ and $a + b = 1$.

Show that, in any group of pupils all of whom got the same mark $M$, there will be (except in one trivial case) perfect negative correlation between scores $X$ and $Y$.

3138. Find the equation of the curve obtained when $y = x^2$ is reflected in the line $y = x - 4$.




3139. In this question, $\mathrm{Av}\big(\mathrm{f}(x)\big)$ is defined as the average value of the function f on the domain $x \in [0, 1]$. Solve for $a$ in $\mathrm{Av}(x^2 - a) = \mathrm{Av}(x^3)$.

3140. A relation between $x$ and $y$ is given implicitly as

$$y^4 = x^4 - x^2.$$

The set of $(x, y)$ points which satisfy this relation is denoted $S$.

(a) Verify that $(0, 0) \in S$.

(b) Show that no other $(x, y) \in S$ satisfies $|x| < 1$.

3141. A random variable has distribution $X \sim B(6, p)$. It is given that $\mathbb{P}(X = 0 \mid X \leq 2) = \frac{4}{31}$. Find $p$.

3142. Solve, for $x, y \in \mathbb{Z}$, the simultaneous equations

$$x^3 + 2y^3 = 24$$
$$x + y = 4.$$

3143. A rogue mathematician says that, if one solution curve $y = \mathrm{f}(x)$ satisfies the differential equation

$$x\frac{dy}{dx} + y = 1,$$

then $y = \mathrm{f}(x) + c$, for $c \in \mathbb{R}$, should also satisfy it. Disprove this claim.

3144. A statistician is devising a test re the proportion of carpenters who work left-handed. In the general population, the proportion of left-handers is 10%, but the statistician suspects that, in groups with reliance on right-handed tools, the proportion who work left-handed is lower.

(a) Write down null and alternative hypotheses for the statistician's suspicion.

(b) At the 1% significance level, determine the smallest sample for which a test of the claim will have a non-zero critical region.

(c) Assuming that, in fact, the null hypothesis holds, determine the probability that, with a sample of size 50, the null hypothesis will be incorrectly rejected.

3145. Prove that the function $\mathrm{h}(x) = \cot^2 3x - \operatorname{cosec}^2 3x$ is constant, and give its value.

3146. A student writes: "Function f has $\mathrm{f}''(k) = 0$, so the graph $y = \mathrm{f}(x)$ has a point of inflection at $\big(k, \mathrm{f}(k)\big)$." Explain the student's error, giving an explicit counterexample.

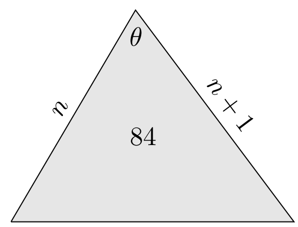
3147. This question involves the binomial expansion of $(1 + x)^{\frac{1}{2}}$, for some $x$ with $|x| < 1$.

(a) Show that $\sqrt{1.08} \approx 1.04$.

(b) Hence, show that $\sqrt{3} \approx \frac{26}{15}$.

3148. Prove that, if a polynomial $\mathrm{f}(x)$ has a factor of $(x - \alpha)^2$, then $\mathrm{f}'(x)$ has a factor of $(x - \alpha)$.

3149. A triangle with sides of length $(n, n+1, n+2)$ has largest interior angle $\theta$ and area 84.




(a) Using the cosine rule and the first Pythagorean trig identity, show that

$$\sin\theta = \frac{\sqrt{3(n^2 + 2n - 3)}}{2n}.$$

(b) Using $A_{\triangle} = \frac{1}{2}ab\sin C$, show that

$$(n + 1)^2(n^2 + 2n - 3) = 37632.$$

(c) Hence, find $n$.

3150. Find the Cartesian equation of the following curve:

$$x = 1 + 5\sec t,$$
$$y = 3 + \tan t.$$

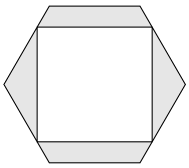
3151. Four values $x_1, x_2, x_3, x_4$ are chosen at random and independently from the interval $[0, 1]$. Find

$$\mathbb{P}(x_1 < x_2 < x_3 < x_4 \mid x_1 < x_4).$$

3152. Show that the curve $y = \sin x$ satisfies

$$\left(\frac{d^3y}{dx^3} + 2\frac{dy}{dx}\right)^2 = (1+y)(1-y).$$

3153. A regular hexagon of side length 1 has a square inscribed. The shapes have parallel sides.



Show that the square has side length $3 - \sqrt{3}$.

3154. A region is defined by the inequalities

$$y \geq 0,$$
$$-3y \geq 4(x-6),$$
$$-3(y-4) \geq -4(x-3).$$

Show that the area of this region is 12.

3155. By differentiating implicitly, prove that normals to the curve $x^2 + y^2 = r^2$, where $r$ is a constant, pass through the origin.

3156. Solve $3^{8x+1} + 2 \times 9^{2x} = 1$.

3157. A tangent is drawn to $y = x^2 e^x$ at $x = 2$. Show that this tangent crosses the $x$ axis at $x = \frac{3}{2}$.

3158. In each case, a list of quantities is given, which refers to a geometric sequence $u_n \in \mathbb{R}$. State, with a reason, whether knowing the quantities listed would allow you to calculate, with certainty, the 99th term of the sequence.

(a) First term; second term.

(b) First term; third term.

(c) First term; fourth term.

3159. A function is defined over the real numbers by $\mathrm{f}(x) = 9x^2 + 6x + 2$. Show that $\mathrm{f}(x) \equiv |\,\mathrm{f}(x)|$.

3160. Verify, by differentiating, that

$$\int \sec\theta \, d\theta = \ln|\sec\theta + \tan\theta| + c.$$

3161. Either prove or disprove the following statement: "For a polynomial function f, $\mathrm{f}(x)$ changes sign in between $x = a$ and $x = b$ if and only if there is a root of f in the interval $(a, b)$."

3162. A particle is modelled as moving with position given by $x = \sin t$, $y = \tan t$.

(a) Sketch the trajectory for $t \in [0, \pi/2)$.

(b) Explain why this model must break down for any particle governed by Newton's laws.

3163. Two cubic functions $\mathrm{f}_1, \mathrm{f}_2$ have different leading coefficients. Their graphs $y = \mathrm{f}_1(x)$ and $y = \mathrm{f}_2(x)$ are tangent at $x = \alpha$, without crossing. Prove that the cubics cross elsewhere.

3164. Find $\mathrm{f}\left(\frac{a+b}{2}\right)$, if f is a linear function such that

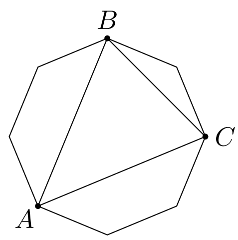
$$\int_a^b \mathrm{f}(x)\, dx = b - a.$$

3165. The random variable $Y$ has a normal distribution. Given that $Y$ is at least one standard deviation from the mean, find the probability that it is at least two standard deviations from the mean.

3166. Solve $\sin^2 x + \frac{\sqrt{3}}{2}\sin 2x = 0$, for $x \in [0, 2\pi)$.

3167. The circles $x^2 + y^2 = 1$ and $\left(x - \frac{1}{2}\right)^2 + \left(y - \frac{1}{2}\right)^2 = k^2$ intersect twice. Find all possible values of $k$, giving your answer in set notation.

3168. Three vertices $A, B, C$ are selected at random from those of a regular octagon, forming triangle $ABC$. An example outcome is shown below:




(a) Show that, if $AB$ is a side of the octagon, then $\angle ACB = 22.5°$.

(b) Find the probability that $\angle ACB$ is acute.

3169. A *rational function* is a function which maps $x$ to a fraction whose numerator and denominator are both polynomials in $x$. Prove that the composition of two rational functions is a rational function.

3170. An object has position $\mathbf{r} = \begin{pmatrix} 1+2e^t \\ 1-e^{2t} \\ 2e^t \end{pmatrix}$ m.

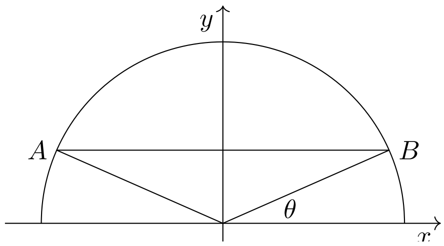
(a) Find the velocity vector.

(b) Determine, in the form $a \ln b$, the exact time at which the object's speed is $\sqrt{8}$ ms$^{-1}$.

3171. An isosceles triangle with integer side lengths has perimeter 50 and area 120. Find its side lengths.

3172. Sketch $y = x^7 + x^2$.

3173. The region inside a unit semicircle $y = \sqrt{1 - x^2}$ is divided into two regions of equal area by the line $y = \sin\theta$, where $\theta$ is a fixed angle.




(a) Show that $4\theta + 2\sin 2\theta = \pi$.

(b) Use fixed-point iteration to find $\theta$ to 4sf.

(c) Establish suitable error bounds for this root.

3174. A differential equation is given as

$$\frac{dy}{dx} + y = 2x + 1.$$

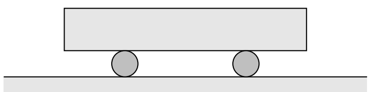
Determine the equation of the linear graph which satisfies the DE.

3175. In a party game, ten pieces of paper, each with a different digit on, are put into a hat. Three guests draw one piece of paper each. Find the probability that

(a) the guests do not pull out any even numbers,

(b) the numbers pulled out make a set of the form $\{n, n+1, n+2\}$.

3176. A uniform block of stone of mass $m$ kg sits on light, cylindrical rollers, as depicted. Initially, the rollers divide the length $l$ of the block in the ratio $1 : 2 : 1$. The block is pushed slowly rightwards, and the rollers do not slip against either the ground or the block.



Determine the maximum possible displacement of the block, if it is not to tip.

3177. Using integration by substitution, show that

$$\int_1^{10} 21x^2\sqrt{5x-1}\,dx = 41816.$$

3178. Show that the lines $y = \pm 2$ are tangent to

$$\left(2y - \sqrt{3}x\right)^2 = 4 - x^2.$$

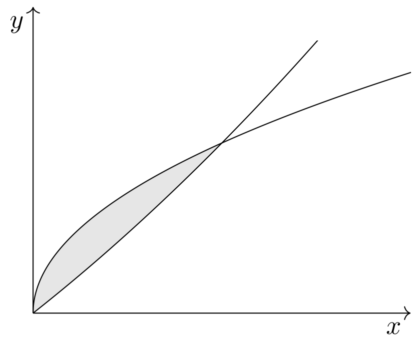
3179. Points $(x, y)$ are coloured red if $\sin(x + y) < 1/2$, and blue if not. Determine the probability that a randomly chosen point is red.

3180. A baseball pitching machine pitches baseballs at 40 $\text{ms}^{-1}$, from 1.5 metres above ground, at angle of inclination $\theta \in [2°, 3°]$ above the horizontal. The batter stands 18 metres from the machine.

Ignoring air resistance, find the set of heights at which the baseballs can reach the batter.

3181. Sketch the curve $y = e^{-x} - e^{-2x}$, labelling all axis intercepts and stationary points.

3182. The diagram below shows a region enclosed by the curves $y = \sqrt{x}$ and $4y = 4x^2 + 7x$.




Find the area of the shaded region.

3183. Show that the sum of the integers from 1 to 1000 which are not multiples of 17 is 471413.

3184. A parametric curve is defined by the equations $x = \sin t$, $y = \sin 2t$, for $t \in [0, 2\pi)$. Determine the coordinates of all points of intersection of this curve with a unit circle centred at the origin.

3185. By differentiating both sides of $x \equiv e^{\ln x}$, prove that the derivative of $\ln x$ is $\frac{1}{x}$.

3186. Show that the following pair of line segments do not intersect:

$$\mathbf{r} = 2t\mathbf{i} + (3 - 2t)\mathbf{j}, \quad t \in [0, 4],$$
$$\mathbf{r} = (3 - t)\mathbf{i} + (1 + 4t)\mathbf{j}, \quad t \in [0, 4].$$

3187. Show that $\dfrac{d^2}{dx^2}(xy) = 2\dfrac{dy}{dx} + x\dfrac{d^2y}{dx^2}$.

3188. Two vertices of an equilateral triangle lie at $(0, 0)$ and $(p, q)$. The third vertex could be at one of two points. Show that the straight line through these points has equation

$$\frac{2y - q}{2x - p} + \frac{p}{q} = 0.$$

3189. One of the following statements, which relate to polynomial functions f and g, is true; the other is not. Prove the one and disprove the other.

(a) If the graph $y = \mathrm{f}(x)$ has a stationary point at $x = \alpha$, then so does the graph $y = \mathrm{fg}(x)$,

(b) If the graph $y = \mathrm{f}(x)$ has a stationary point at $x = \alpha$, then so does the graph $y = \mathrm{gf}(x)$.

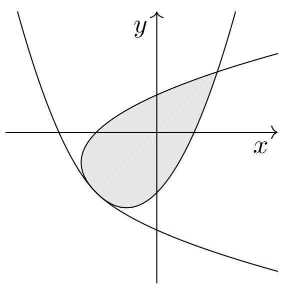
3190. Show that $\lim_{x \to e} \dfrac{1 - \ln x}{(\ln x)^2 - 1} = -\frac{1}{2}$.

3191. State, giving a reason, which of the implications $\Longrightarrow$ , $\Longleftarrow$ , $\Longleftrightarrow$ links the following statements concerning polynomial functions f and g:

① $\mathrm{f}(x) \equiv \mathrm{g}(x)$,

② $\mathrm{f}'(x) \equiv \mathrm{g}'(x)$.

3192. The diagram shows a region enclosed by the curves $y + 1 = x^2 + x$ and $x + 1 = y^2 + y$:




Determine the area of the region enclosed.

3193. Show that the range of $x \mapsto x^6 + x^3 + 1$, over the real numbers, is $\{y \in \mathbb{R} : y \geq 3/4\}$.

3194. Show that there are infinitely many $(x, y, z)$ points which satisfy the simultaneous equations

$$\begin{aligned} x + y + z &= 1, \\ 2x - y + z &= 3, \\ 3x - 3y + z &= 5. \end{aligned}$$

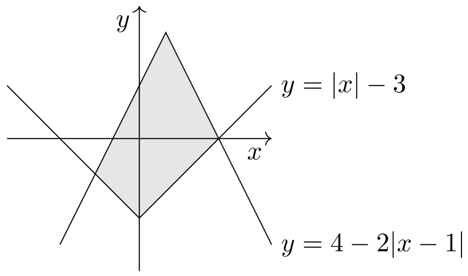
3195. From a large population, a sample of four is taken. Find the probability that at least two of the four are above the 90th percentile for some statistic.

3196. Let $a$ and $b$ be prime numbers. Prove that $a^2b^3$ is not a perfect square.

3197. A polynomial function f has three fixed points. By considering points for which distance between $y = \mathrm{f}(x)$ and $y = x$ is locally maximised, show that there are at least two $x$ values for which $\mathrm{f}'(x) = 1$.

3198. Express $-8x^3 + 16x^2 - 10x - 2$ in simplified terms of the variable $u = 1 - 2x$.

3199. Two graphs are represented below:




Show that the area of the shaded region is $\frac{47}{3}$.

3200. A function f, whose domain is $\mathbb{R}$, has the following property: when a tangent $T$ is drawn to the graph $y = \mathrm{f}(x)$ at any point $x = x_0$, the angle $\alpha$ between $T$ and the $x$ axis is given by $\alpha = \arctan(\cos x_0)$. Find all functions f with this property.

—— End of 32nd Hundred ——